ریاضیات عالی پیشرفته

Numerical Methods for Engineers

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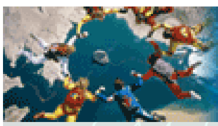

Numerical Methods for Engineers

Steven C. Chapra
Raymond P. Canale
Fifth Edition

~ Linear Algebraic Equations ~

Gauss Elimination

Chapter 9

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Solving Systems of Equations

- A linear equation in n variables:

$$a_1x_1 + a_2x_2 + \ \dots + a_nx_n = b$$

- For small ($n \leq 3$), linear algebra provides several tools to solve such systems of linear equations:
  - Graphical method
  - Cramer's rule
  - Method of elimination
- Nowadays, easy access to computers makes the solution of **very large** sets of linear algebraic equations possible

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Determinants and Cramer's Rule

$$[A]\{x\} = \{b\}$$

[A] : ***coefficient matrix***

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{D} \qquad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}}{D} \qquad x_3 = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{D}$$

D : Determinant of A matrix

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Computing the Determinant

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$[A]\{x\} = \{B\}$$

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$D_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{32}a_{23}$$

$$D_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}a_{33} - a_{31}a_{23}$$

$$D_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}a_{32} - a_{31}a_{22}$$

$$\text{Determinant of A} = D = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

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Gauss Elimination

- Solve $Ax = b$
- Consists of two phases:
  - **Forward elimination**
  - **Back substitution**
- *Forward Elimination* reduces $Ax = b$ to an upper triangular system $Tx = b'$
- *Back substitution* can then solve $Tx = b'$ for x

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a'_{22} & a'_{23} & b'_2 \\ 0 & 0 & a''_{33} & b''_3 \end{array}\right]$$

Forward Elimination

$$\Downarrow$$

$$x_3 = \frac{b''_3}{a''_{33}} \qquad x_2 = \frac{b'_2 - a'_{23}x_3}{a'_{22}}$$

$$x_1 = \frac{b_1 - a_{13}x_3 - a_{12}x_2}{a_{11}}$$

Back Substitution

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Forward Elimination

-(3/1)
-(2/1)

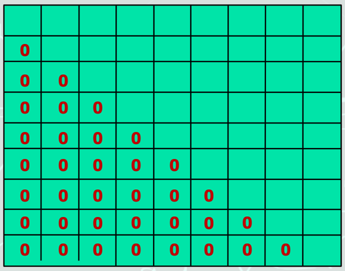
$x_1 - x_2 + x_3 = 6$
$3x_1 + 4x_2 + 2x_3 = 9$
$2x_1 + x_2 + x_3 = 7$

-(3/7)

$x_1 - x_2 + x_3 = 6$
$0 + 7x_2 - x_3 = -9$
$0 + 3x_2 - x_3 = -5$

$x_1 - x_2 + x_3 = 6$
$0 \quad 7x_2 - x_3 = -9$
$0 \quad 0 \quad -(4/7)x_3 = -(8/7)$

Solve using BACK SUBSTITUTION: $x_3 = 2$ $x_2 = -1$ $x_1 = 3$



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Back Substitution

$$
\begin{array}{cccccc}
1x_0 & +1x_1 & -1x_2 & +4x_3 & = & 8 \\
 & -2x_1 & -3x_2 & +1x_3 & = & 5 \\
 & & 2x_2 & -3x_3 & = & 0 \\
 & & & 2x_3 & = & 4
\end{array}
$$

$x_3 = 2$

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Back Substitution

$1x_0$	$+1x_1$	$-1x_2$	$=$	0
	$-\ 2x_1$	$-3x_2$	$=$	3
$\boldsymbol{x_2 = 3}$		$2x_2$	$=$	6

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Back Substitution

$1x_0$	$+1x_1$	$=$	3
$\boldsymbol{x_1 = -6}$	$-\ 2x_1$	$=$	12

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Back Substitution

$x_0 = 9$

$1x_0 = 9$

Back Substitution
(* Pseudocode *)

```
for i ← n down to 1 do

    /* calculate x_i */
    x [ i ] ← b [ i ] / a [ i, i ]

    /* substitute in the equations above */
    for  j ← 1 to i-1  do
        b [ j ] ← b [ j ] – x [ i ] × a [ j, i ]
    endfor
endfor
```

Time Complexity? ➔➔ $O(n^2)$

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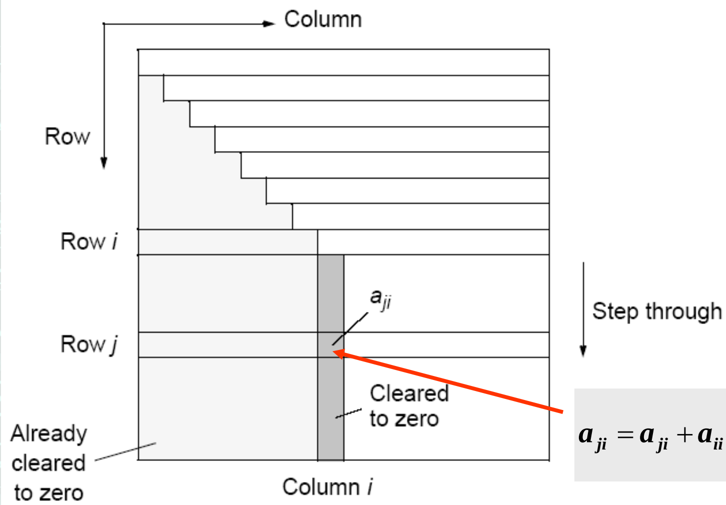
Forward Elimination



$$a_{ji} = a_{ji} + a_{ii}\left(\frac{-a_{ji}}{a_{ii}}\right) = 0$$

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Forward Elimination

MULTIPLIERS

	$4x_0$	$+6x_1$	$+2x_2$	$-2x_3$	$=$	8
-(2/4)	$2x_0$		$+5x_2$	$-2x_3$	$=$	4
-(-4/4)	$-4x_0$	$-3x_1$	$-5x_2$	$+4x_3$	$=$	1
-(8/4)	$8x_0$	$+18x_1$	$-2x_2$	$+3x_3$	$=$	40

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Forward Elimination

M U L T I P L I E R S

$$\begin{array}{ccccccc} 4x_0 & +6x_1 & +2x_2 & -2x_3 & = & 8 \\ & -3x_1 & +4x_2 & -1x_3 & = & 0 \\ & +3x_1 & -3x_2 & +2x_3 & = & 9 \\ & +6x_1 & -6x_2 & +7x_3 & = & 24 \end{array}$$

-(3/-3)

-(6/-3)

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Forward Elimination

M U L T I P L I E R

$$\begin{array}{ccccccc} 4x_0 & +6x_1 & +2x_2 & -2x_3 & = & 8 \\ & -3x_1 & +4x_2 & -1x_3 & = & 0 \\ & & 1x_2 & +1x_3 & = & 9 \\ & & 2x_2 & +5x_3 & = & 24 \end{array}$$

??

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Forward Elimination

$4x_0$	$+6x_1$	$+2x_2$	$-2x_3$	$=$	8
	$-3x_1$	$+4x_2$	$-1x_3$	$=$	0
		$1x_2$	$+1x_3$	$=$	9
			$3x_3$	$=$	6

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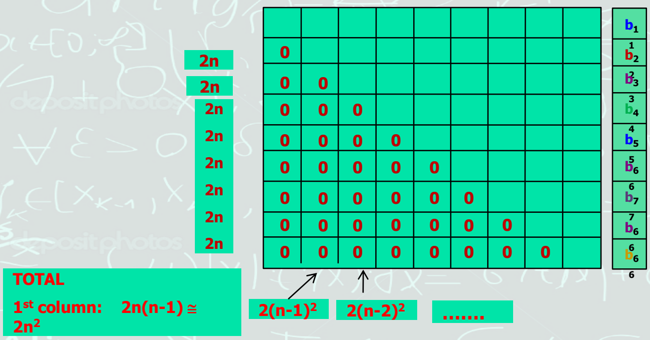
Operation count in Forward Elimination



TOTAL # of Operations for FORWARD ELIMINATION :

$$2n^2 + 2(n-1)^2 + \ldots + 2*(2)^2 + 2*(1)^2 = 2\sum_{i=1}^{n} i^2$$

$$= 2\frac{n(n+1)(2n+1)}{6}$$

$$= O(n^3)$$

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**herePitfalls of Elimination Methods

Division by zero

It is possible that during both elimination and back-substitution phases a division by zero can occur.

For example:

$$2x_2 + 3x_3 = 8$$
$$4x_1 + 6x_2 + 7x_3 = -3$$
$$2x_1 + x_2 + 6x_3 = 5$$

$$A = \begin{array}{ccc} 0 & 2 & 3 \\ 4 & 6 & 7 \\ 2 & 1 & 6 \end{array}$$

Solution: ***pivoting*** (to be discussed later)

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Pitfalls (cont.)

Round-off errors

- Because computers carry only a limited number of significant figures, round-off errors will occur and they will *propagate* from one iteration to the next.
- This problem is especially important when **large** numbers of equations (100 or more) are to be solved.
- Always use **double-precision** numbers/arithmetic. It is slow but needed for correctness!
- It is also a good idea to substitute your results back into the original equations and check whether a substantial error has occurred.

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Pitfalls (cont.)

ill-conditioned systems - small changes in coefficients result in large changes in the solution. Alternatively, a wide range of answers can approximately satisfy the equations.

(*Well-conditioned systems* – small changes in coefficients result in small changes in the solution)

Problem: Since *round off errors* can induce small changes in the coefficients, these changes can lead to large solution errors in *ill-conditioned* systems.

Example:

$x_1 + 2x_2 = 10$

$\mathbf{1.1}x_1 + 2x_2 = 10.4$

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{D} = \frac{\begin{vmatrix} 10 & 2 \\ 10.4 & 2 \end{vmatrix}}{1(2) - 2(1.1)} = \frac{2(10) - 2(10.4)}{-0.2} = 4 \qquad x_2 = 3$$

$x_1 + 2x_2 = 10$

$\mathbf{1.05}x_1 + 2x_2 = 10.4$

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{D} = \frac{\begin{vmatrix} 10 & 2 \\ 10.4 & 2 \end{vmatrix}}{1(2) - 2(1.05)} = \frac{2(10) - 2(10.4)}{-0.1} = 8 \qquad x_2 = 1$$

Pitfalls (cont.)

ill-conditioned systems (cont.) –

- Surprisingly, **substitution** of the erroneous values, x_1=8 and x_2=1, into the original equation **will not** reveal their incorrect nature clearly:

$x_1 + 2x_2 = 10$	8+2(1) = 10	(the same!)
$1.1x_1 + 2x_2 = 10.4$	1.1(8)+2(1)=10.8	(close!)

IMPORTANT CONCLUSION:
An ill-conditioned system is one with a *determinant* close to **zero**

- If determinant D=0 then there are infinitely many solutions ➔ ***singular system***
- ***Scaling*** (multiplying the coefficients with the same value) does not change the equations but changes the value of the determinant in a significant way.
  However, it does not change the *ill-conditioned* state of the equations!

  DANGER! It may hide the fact that the system is ill-conditioned!!

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How can we find out whether a system is *ill-conditioned* or not?
Not easy! Luckily, most engineering systems yield well-conditioned results!

Is the system *ill-conditioned*?

- One way to find out: First **scale (normalize)** each row such that no coefficient is larger than 1. Then compute the determinant and check if it is close to zero.
- Another way: change the coefficients slightly and recompute & compare

COMPUTING THE DETERMINANT:

- Given an upper triangular sys. of equations

$D = t_{11} t_{22} t_{33}$

In general, $D = t_{11} t_{22} \ldots t_{nn}$

$$D = \begin{vmatrix} t_{11} & t_{12} & t_{13} \\ 0 & t_{22} & t_{23} \\ 0 & 0 & t_{33} \end{vmatrix}$$

If ***pivoting*** is used then

$\boldsymbol{D = t_{11} t_{22} \ldots t_{nn} (-1)^p}$ *where p is the number of times the rows are pivoted*

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Techniques for Improving Solutions

- Use of more significant figures – ***double precision*** *arithmetic*

- ***Pivoting***

  If a pivot element is zero, normalization step leads to division by zero. The same problem may arise, when the pivot element is close to zero. Problem can be avoided:

  - ***Partial pivoting***

    Switching the rows below so that the largest element is the pivot element.

    Go over the solution in: CHAP9e-Problem-11.doc

  - ***Complete pivoting***
    - Searching for the largest element in all rows and columns then switching.
    - This is rarely used because switching columns changes the order of x's and adds significant complexity and overhead ➔ costly

- ***Scaling***
  - used to reduce the round-off errors and improve accuracy

Gauss-Jordan Elimination

a_{11}	0	0	0	0	0	0	0	0	b_{11}
0	x_{22}	0	0	0	0	0	0	0	b_{22}
0	0	x_{33}	0	0	0	0	0	0	b_{33}
0	0	0	x_{44}	0	0	0	0	0	b_{44}
0	0	0	0	x_{55}	0	0	0	0	b_{55}
0	0	0	0	0	x_{66}	0	0	0	b_{66}
0	0	0	0	0	0	x_{77}	0	0	b_{77}
0	0	0	0	0	0	0	x_{88}	0	b_{66}
0	0	0	0	0	0	0	0	x_{99}	b_{66}

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Gauss-Jordan Elimination: **Example**

$$\begin{bmatrix} 1 & 1 & 2 \\ -1 & -2 & 3 \\ 3 & 7 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 10 \end{bmatrix} \qquad \text{Augmented Matrix:} \begin{bmatrix} 1 & 1 & 2 & | & 8 \\ -1 & -2 & 3 & | & 1 \\ 3 & 7 & 4 & | & 10 \end{bmatrix}$$

R2 ← R2 - (-1)R1

R3 ← R3 - (3)R1

$$\begin{bmatrix} 1 & 1 & 2 & | & 8 \\ 0 & -1 & 5 & | & 9 \\ 0 & 4 & -2 & | & -14 \end{bmatrix}$$

Scaling R2:

R2 ← R2/(**-1**)

$$\begin{bmatrix} 1 & 1 & 2 & | & 8 \\ 0 & 1 & -5 & | & -9 \\ 0 & 4 & -2 & | & -14 \end{bmatrix}$$

R1 ← R1 - (1)R2

R3 ← R3-(4)R2

$$\begin{bmatrix} 1 & 0 & 7 & | & 17 \\ 0 & 1 & -5 & | & -9 \\ 0 & 0 & 18 & | & 22 \end{bmatrix}$$

Scaling R3:

R3 ← R3/(**18**)

$$\begin{bmatrix} 1 & 0 & 7 & | & 17 \\ 0 & 1 & -5 & | & -9 \\ 0 & 0 & 1 & | & 11/9 \end{bmatrix}$$

R1 ← R1 - (7)R3

R2 ← R2-(-5)R3

$$\begin{bmatrix} 1 & 0 & 0 & | & 8.444 \\ 0 & 1 & 0 & | & -2.888 \\ 0 & 0 & 1 & | & 1.222 \end{bmatrix}$$

RESULT:

x_1=8.45, x_2=-2.89, x_3=1.23

Time Complexity? ➔➔ ***O(n^3)***

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LU Decomposition and Matrix Inversion

Chapter 10

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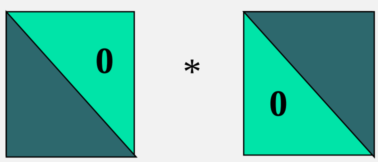
Solve $\boldsymbol{A} \cdot \boldsymbol{x} = \boldsymbol{b}$ *(system of linear equations)*

Decompose $\boldsymbol{A} = \boldsymbol{L} \cdot \boldsymbol{U}$



L : Lower Triangular Matrix U : Upper Triangular Matrix

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To solve $[A]\{x\}=\{b\}$

$[L][U]=[A] \Rightarrow [L][U]\{x\}=\{b\}$

Consider $[U]\{x\}=\{d\}$
$[L]\{d\}=\{b\}$

1. Solve $[L]\{d\}=\{b\}$ using **forward substitution** to get $\{d\}$
2. Use **back substitution** to solve $[U]\{x\}=\{d\}$ to get $\{x\}$

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$$[A]\{x\}=\{b\} \Rightarrow [L][U]\{x\}=\{b\}$$

$$[A]=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

[*L*] [*U*]

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$$[A]\{x\}=\{b\} \Rightarrow [L][U]\{x\}=\{b\}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Gauss Elimination ➔ ➔

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a''_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} b_1 \\ b'_2 \\ b''_3 \end{bmatrix}$$

[U]

$$[L]=\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

Coefficients used during the elimination step

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$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a''_{33} \end{bmatrix}$$

$$l_{21}=\frac{a_{21}}{a_{11}}$$

[$L \cdot U$]

$$l_{31}=\frac{a_{31}}{a_{11}}$$

$$l_{32}=?$$

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Example: $A = L \cdot U$

$$\begin{bmatrix} -1 & 2.5 & 5 \\ -2 & 9 & 11 \\ 4 & -22 & -20 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -4 & -3 & 1 \end{bmatrix}\begin{bmatrix} -1 & 2.5 & 5 \\ 0 & 4 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

Gauss Elimination

Coefficients

l_{21} = -2/-1= 2

l_{31} = 4/-1= -4

$$\begin{bmatrix} -1 & 2.5 & 5 \\ -2 & 9 & 11 \\ 4 & -22 & -20 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 2.5 & 5 \\ 0 & 4 & 1 \\ 0 & -12 & 0 \end{bmatrix}$$

Coefficients

l_{32} = -12/4= -3

$$\begin{bmatrix} -1 & 2.5 & 5 \\ 0 & 4 & 1 \\ 0 & -12 & 0 \end{bmatrix} \Rightarrow \boxed{\begin{bmatrix} -1 & 2.5 & 5 \\ 0 & 4 & 1 \\ 0 & 0 & 3 \end{bmatrix}}$$

[L]

[U]

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MATRIX INVERSE

$A.\ A^{-1} = I$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solve in n=3 major steps

1

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

2

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

3

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Solve each one
using A=L·U method ➔ e.g.

$$LU \cdot \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Solve Problem 10.6. Solution file is available on the web.

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Special Matrices and Gauss-Seidel

Chapter 11

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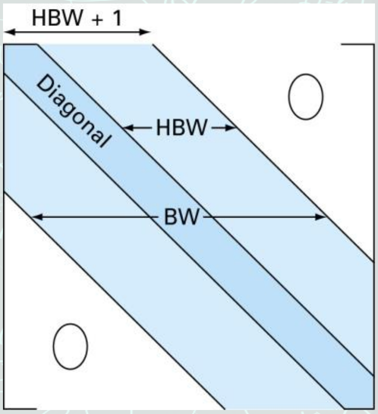
- Certain matrices have particular structures that can be exploited to develop efficient solution schemes (e.g. *banded*, *symmetric*)
- A *banded matrix* is a square matrix that has all elements equal to zero, with the exception of a **band** centered on the main diagonal.
- Standard Gauss Elimination is inefficient in solving banded equations because unnecessary space and time would be expended on the storage and manipulation of **zeros**.
- There is no need to store or process the zeros (off of the band)



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Solving Tridiagonal Systems (Thomas Algorithm)

A tridiagonal system has a bandwidth of 3

$$\begin{bmatrix} f_1 & g_1 & & \\ e_2 & f_2 & g_2 & \\ & e_3 & f_3 & g_3 \\ & & e_4 & f_4 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{Bmatrix}$$

$$A = L * U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ e'_2 & 1 & 0 & 0 \\ 0 & e'_3 & 1 & 0 \\ 0 & 0 & e'_4 & 1 \end{bmatrix} \begin{bmatrix} f_1 & g_1 & & \\ & f'_2 & g_2 & \\ & & f'_3 & g_3 \\ & & & f'_4 \end{bmatrix}$$

DECOMPOSITION

DO $k = 2, n$

$e_k = e_k / f_{k-1}$

$f_k = f_k - e_k \, g_{k-1}$

END DO

Time Complexity? **O(n)**

vs. **O(n³)**

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Tridiagonal Systems (cont.)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ e'_2 & 1 & 0 & 0 \\ 0 & e'_3 & 1 & 0 \\ 0 & 0 & e'_4 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} f_1 & g_1 & & \\ & f'_2 & g_2 & \\ & & f'_3 & g_3 \\ & & & f'_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{\{d\}} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ e'_2 & 1 & 0 & 0 \\ 0 & e'_3 & 1 & 0 \\ 0 & 0 & e'_4 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix}$$

$$\begin{bmatrix} f_1 & g_1 & & \\ & f'_2 & g_2 & \\ & & f'_3 & g_3 \\ & & & f'_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix}$$

Forward Substitution

$d_1 = r_1$

DO $k = 2, n$

$d_k = r_k - e_k \, d_{k-1}$

END DO

Back Substitution

$x_n = d_n / f_n$

DO $k = n-1, 1, -1$

$x_k = (d_k - g_k \cdot x_{k+1}) / f_k$

END DO

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Cholesky Decomposition
(for ***Symmetric Positive Definite***† Matrices)

† A *positive definite matrix* is one for which the product $\{X\}^T[A]\{X\}$ is greater than zero for all nonzero vectors X

$$[A]=[A]^T=\begin{bmatrix} a_{11} & a_{21} & a_{31} & a_{41} \\ a_{21} & a_{22} & a_{32} & a_{42} \\ a_{31} & a_{32} & a_{33} & a_{43} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

$$[A]=[A]^T=L*L^T=\begin{bmatrix} l_{11} & & & \\ l_{21} & l_{22} & & \\ l_{31} & l_{32} & l_{33} & \\ l_{41} & l_{42} & l_{43} & l_{44} \end{bmatrix}\begin{bmatrix} l_{11} & l_{21} & l_{31} & l_{41} \\ & l_{22} & l_{32} & l_{42} \\ & & l_{33} & l_{43} \\ & & & l_{44} \end{bmatrix}$$

L^T means **Transpose** of L

$$l_{ki}=\frac{a_{ki}-\sum_{j=1}^{i-1} l_{ij}l_{kj}}{l_{ii}} \quad \text{for } k=1,2,\dots,n \quad i=1,2,\dots,k-1 \qquad l_{kk}=\sqrt{a_{kk}-\sum_{j=1}^{k-1} l_{kj}^2}$$

Time Complexity:
$O(n^3)$ but requires half the number of operations as standard Gaussian elimination.

**here Jacobi Iterative Method

Iterative methods provide an **alternative** to the *elimination methods*.

$$\mathbf{Ax}=\mathbf{b} \qquad \mathbf{A}=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \mathbf{D}=\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

$$[\mathbf{D}+(\mathbf{A}-\mathbf{D})]\mathbf{x}=\mathbf{b} \quad \Rightarrow \quad \mathbf{Dx}=\mathbf{b}-(\mathbf{A}-\mathbf{D})\mathbf{x} \quad \Rightarrow \quad \mathbf{x}=\mathbf{D}^{-1}[\mathbf{b}-(\mathbf{A}-\mathbf{D})\mathbf{x}]$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 1/a_{11} & 0 & 0 \\ 0 & 1/a_{22} & 0 \\ 0 & 0 & 1/a_{33} \end{bmatrix}*\left(\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}-\begin{bmatrix} 0 & a_{12} & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & a_{32} & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right)$$

$$x_1^k=\frac{b_1-a_{12}x_2^{k-1}-a_{13}x_3^{k-1}}{a_{11}} \qquad x_2^k=\frac{b_2-a_{21}x_1^{k-1}-a_{23}x_3^{k-1}}{a_{22}} \qquad x_3^k=\frac{b_3-a_{31}x_1^{k-1}-a_{32}x_2^{k-1}}{a_{33}}$$

Choose an initial guess (i.e. all zeros) and Iterate until the equality is satisfied.
No guarantee for convergence! Each iteration takes $O(n^2)$ time!

Gauss-Seidel

- The *Gauss-Seidel* method is a commonly used *iterative method*.
- It is same as **Jacobi technique** except with one important difference:
  A newly computed x value (say x_k) is substituted in the subsequent equations (equations $k+1, k+2, \ldots, n$) **in the same iteration**.

Example: Consider the *3x3* system below:

$$x_1^{new} = \frac{b_1 - a_{12}x_2^{old} - a_{13}x_3^{old}}{a_{11}}$$

$$x_2^{new} = \frac{b_2 - a_{21}x_1^{new} - a_{23}x_3^{old}}{a_{22}}$$

$$x_3^{new} = \frac{b_3 - a_{31}x_1^{new} - a_{32}x_2^{new}}{a_{33}}$$

$$\{X\}_{old} \leftarrow \{X\}_{new}$$

- First, choose initial guesses for the x's.
- A simple way to obtain initial guesses is to assume that they are all **zero**.
- Compute **new** $\mathbf{x_1}$ using the previous iteration values.
- **New** $\boldsymbol{x_1}$ is substituted in the equations to calculate x_2 and x_3
- The process is repeated for $\mathbf{x_2}$, $\mathbf{x_3}$, …

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Convergence Criterion for Gauss-Seidel Method

- Iterations are repeated until the convergence criterion is satisfied:

$$\left|\varepsilon_{a,i}\right| = \left|\frac{x_i^j - x_i^{j-1}}{x_i^j}\right| 100\% \prec \varepsilon_s$$

For all i, where j and j-1 are the *current* and *previous* iterations.

- As any other iterative method, the **Gauss-Seidel** method has problems:
  - It may not converge or it converges very slowly.
- If the coefficient matrix A is **Diagonally Dominant** Gauss-Seidel is guaranteed to converge.

Diagonally Dominant ➔ **For each equation i :**

$$\left|a_{ii}\right| \succ \sum_{\substack{j=1 \\ j \neq i}}^{n} \left|a_{i,j}\right|$$

- Note that this is not a necessary condition, i.e. the system *may* still have a chance to converge even if A is not diagonally dominant.

Time Complexity: Each iteration takes O(n^2)